$\mathbb{S}$

Surreal Geometry

Given the empty set $\{\}$ (or not!); the operations union $\cup$, intersection $\cap$, and subtraction $\setminus$; and the relationships membership $\in$, contained in $\subseteq$, and set equality $\equiv$; $\mathbb{S}$ is the *surreal geometry*, such that:

1. Genesis

Given $x = \{\mathcal{L}[x] \mid \mathcal{R}[x]\}$, $x \in \mathbb{S}$ if and only if:

- The bounds $\mathcal{L}[x]$ and $\mathcal{R}[x]$ are two sets whose elements are other members of $\mathbb{S}$ or sets of their bounds
- There is not $x_l \in \mathcal{L}_D[x]$ and $x_r \in \mathcal{R}_D[x]$ such that $x_r \leq x_l$ for any $D \in \mathbb{D}$

where, given also $y \in \mathbb{S}$, $x \leq y$ if and only if:

- There is not $x_l \in \mathcal{L}_D[x]$ such that $y_D \leq x_l$ and not $y_r \in \mathcal{R}_D[y]$ such that $y_r \leq x_D$ for any $D \in \mathbb{D}$

and where, for all $x \in \mathbb{S}$, $X \subseteq \mathbb{S}$, and $\mathcal{B} \in \{\mathcal{L}, \mathcal{R}\}$, and with $D \in \mathbb{D}$:

- $X_D \equiv \{x_D \text{ for all } x \in X\} \setminus \{0 \text{ such that } 0 \notin X \text{ and } D \not\equiv \{\}\}$ and $x_D = \{\mathcal{L}_D[x] \mid \mathcal{R}_D[x]\}$
- $\mathcal{B}_{\{D\}} \equiv (\mathcal{B} \cap \mathbb{S})_{\{D\}} \cup \{b \text{ for all } b \in B_D \text{ such that } B \in \mathcal{B} \setminus \mathbb{S}\}$ and $\mathcal{B}_{\{\}} \equiv (\mathcal{B} \cap \mathbb{S})_{\{\}}$

Additionally:

- $x = y$ if and only if $x \leq y$ and $y \leq x$; and $x < y$ and $y > x$ if and only if $x \leq y$ and $y \not\leq x$
- The simplest $x \in \mathbb{S}$ is $\{|\}$, or 0; $\{0 \mid\}$ is 1. $\mathbb{N}$ is defined such that $0 \in \mathbb{N}$, and $\{n \mid\} \in \mathbb{N}$ if $n \in \mathbb{N}$
- $\mathbb{D}$ is defined such that $\{\} \in \mathbb{D}$, and $\{D\} \in \mathbb{D}$ if $D \in \mathbb{D}$

2. Algebra

Given $x \in \mathbb{S}$, $X \subseteq \mathbb{S}$, $n \in \mathbb{N}$, and $Z_{n+1} \equiv \{Z_n\}$ with $Z_0 \equiv \{\}$:

- $x_n = \langle x_{n-1}, x_{-n} \rangle_n$ if $n > 0$ and $x_{-n} = x_{Z_n}$
- $X_n \equiv \{x_n \text{ for all } x \in X\}$ and $X_{-n} \equiv \{x_{-n} \text{ for all } x \in X\}$

where, given $x_\circ, x_\bullet \in \mathbb{S}_n$:

- $\langle x_\circ, x_\bullet \rangle_{n+1} = \{A_\mathcal{L} \mid A_\mathcal{R}\}$ with $A_\mathcal{B} \equiv \mathcal{B}[x_\circ] \cup \{\mathcal{B}[x_\bullet] \cup Z_n\}$ for $\mathcal{B} \in \{\mathcal{L}, \mathcal{R}\}$

and, given also $y_\circ, y_\bullet \in \mathbb{S}_n$, and $x, y \in \mathbb{S}_{n+1}$ such that $x = \langle x_\circ, x_\bullet \rangle_{n+1}$ and $y = \langle y_\circ, y_\bullet \rangle_{n+1}$:

- $x^* = \langle x_\circ^*, -x_\bullet^* \rangle_{n+1}$ with $x_0^* = x_0$, and $-x = \{-\mathcal{R}[x] \mid -\mathcal{L}[x]\}$
- $x + y = \langle x_\circ + y_\circ, x_\bullet + y_\bullet \rangle_{n+1}$, and $x_\circ - y_\circ = x_\circ + (-y_\circ)$, where $x_0 + y_0 = \{S_\mathcal{L} \mid S_\mathcal{R}\}$ and $S_\mathcal{B} \equiv (\mathcal{B}[x_0] + y_0) \cup (\mathcal{B}[y_0] + x_0)$ for $\mathcal{B} \in \{\mathcal{L}, \mathcal{R}\}$
- $xy = \langle x_\circ y_\circ + x_\bullet^* y_\bullet, x_\bullet y_\circ + x_\circ^* y_\bullet \rangle_{n+1}$ where $x_0 y_0 = \{P_\mathcal{L}^\mathcal{L} \cup P_\mathcal{R}^\mathcal{R} \mid P_\mathcal{L}^\mathcal{R} \cup P_\mathcal{R}^\mathcal{L}\}$ and $P_\mathcal{B}^\mathcal{C} \equiv \mathcal{B}[x_0]y_0 + x_0\mathcal{C}[y_0] - \mathcal{B}[x_0]\mathcal{C}[y_0]$ for $\mathcal{B}, \mathcal{C} \in \{\mathcal{L}, \mathcal{R}\}$
- If $x_\circ x_\circ^* \in \mathbb{S}_0 \setminus \{0\}$, $x_\circ^{-1} = x_\circ^*(x_\circ x_\circ^*)^{-1}$, and $x_0^{-1} = \{\{0\} \cup I_\mathcal{L}^\mathcal{R} \cup I_\mathcal{R}^\mathcal{L} \mid I_\mathcal{L}^\mathcal{L} \cup I_\mathcal{R}^\mathcal{R}\}$ if $x_0 > 0$ where $I_\mathcal{B}^\mathcal{C} \equiv \left(1 + (\mathcal{C}[x_0] - x_0)\mathcal{B}[x_0^{-1}]\right)\mathcal{C}[x_0]^{-1}$ for $\mathcal{B}, \mathcal{C} \in \{\mathcal{L}, \mathcal{R}\}$

Operations are applied element-wise over sets, combining all pairs for operations over two sets.

3. Calculus

Given a function $f[x]$ that maps every $x \in X$ such that $X \subseteq \mathbb{S}_n$ into a unique element of $\mathbb{S}$, the infinitesimal $\varepsilon = \left\{0 \mid (\mathbb{N} \setminus \{0\})^{-1}\right\}$, and the function $\rho[x]$ such that:

- $\rho[x] = \langle \rho[x_{n-1}], \rho[x_{-n}] \rangle_n$ if $n > 0$, and $\rho[X] \equiv \{\rho[x] \text{ for all } x \in X\}$
- $\rho[x_0] = \{x_l \in \mathbb{Q} \mid x_r \in \mathbb{Q}\}$ such that $x_l < x_0$ and $x_0 < x_r$ with $\mathbb{Q} \equiv (\mathbb{N} \cup -\mathbb{N})\,(\mathbb{N} \setminus \{0\})^{-1}$

there exist $f^\nabla[x]$ for $x \in X$ and $f^{\nabla\Delta}[X]$ such that:

- $f^\nabla[x] = \left\langle f^\nabla[x_{n-1}], f^\nabla[x_{-n}] \right\rangle_n$ if $n > 0$, and $f^\nabla[x_0] = \rho\left[(f[x_0 + \varepsilon] - f[x_0])\,\varepsilon^{-1}\right]$
- $f^{\nabla\Delta}[X] = \rho[f^\Delta[X^\Delta] - f^\Delta[X^\nabla]]$ where $f^\Delta[X] = f[x_\rho]$ if $\rho[X] \equiv \{x_\rho\}$ with $x_\rho \in \mathbb{S}$

where $X^\Delta \equiv X_\varepsilon \setminus X$ and $X^\nabla \equiv X \setminus X_\varepsilon$ with $X_\varepsilon \equiv X + \varepsilon\delta[X]$ such that $\delta[X] = \langle \delta[X_{n-1}], \delta[X_{-n}] \rangle_n$ and $\delta[X_0] = 1$ if there exists $x, y \in X_0$ such that $x \neq y$ and 0 otherwise.

4. Physics

Given the physical universe with entropy function $S[t] \in \mathbb{S}_0$ for a time t, every change in the space-time $X \subseteq \mathbb{S}_n$ is defined by an energy field $E[x]$ with $E^\Delta[X] \in \mathbb{S}_0$ such that:

- There is not $E_<[x]$ with $E_<[x] = E[x]$ for all $x \in X^\Delta \cup X^\nabla$ and $E_<^\Delta[X] < E^\Delta[X]$
- $S[t^\nabla] \leq S[t^\Delta]$ for all $t^\nabla \in X_0^\nabla$ and $t^\Delta \in X_0^\Delta$

And, therefore:

- If $P[x]$ is the wave function for a particle of mass m, $P^\nabla[x]j = m(P[x])^*$ with $j \in \mathbb{S}$ and $jj = -1$
- If $F[x]$ is the electromagnetic field, the charge and current density is $J[x] = F^\nabla[x]$
- If x represents the momentum of a particle with energy x_0 and mass m, then $xx^* = mm$. At larger scales, if $G[x]$ is the curvature of the gravitational field, the flow of energy and matter is $T[x] = G[x]$

The rest of physics is left as an exercise for the reader.

References

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Appendix

A. Properties. Given $x, y, z \in \mathbb{S}$, it can be proven that:

- $x + 0 = x$, $x + y = y + x$ and $x + (y + z) = (x + y) + z$
- $1x = x$, $x_0 y = y x_0$ and $x(yz) = (xy)z$ (also, $(xy)^* = x^* y^*$)
- $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$

B. Base. Given $\mathbb{S}^0 \equiv \mathbb{S}_0$, $\mathbb{S}^{n+1} \equiv \{x \in S^m \text{ with } S^m \equiv \langle S^{m-1}, \mathbb{S}^n \rangle_m,\ S^n \equiv \{0\} \text{ and } m > n\}$ and:

	$\mathbb{S}^0$	$\mathbb{S}^1$	$\mathbb{S}^2$	$\mathbb{S}^3$
$\mathbb{S}_0$	$1 = \{0 \mid\}$			
$\mathbb{S}_1$		$e_1 = \{\{0\} \mid\}$		
$\mathbb{S}_2$		$e_2 = \{\{0, \{\}\} \mid\}$	$e_{2,1} = \{\{\{0\}\} \mid\}$	
$\mathbb{S}_3$		$e_3 = \{\{0, \{\{\}\}\} \mid\}$	$e_{3,2} = \{\{\{0, \{\}\}\} \mid\}$ $e_{3,1} = \{\{\{0\}, \{\{\}\}\} \mid\}$	$e_{3,2,1} = \{\{\{\{0\}\}\} \mid\}$

$e_{ij} \in \mathbb{S}_i \cap \mathbb{S}^j$ for each e_{ij} in row i and column j and, given $n, m \in \mathbb{N}$ with $0 < m$ and $m < n$:

- $e_m = \langle m_\circ, m_\bullet \rangle_n = \langle e_m, 0 \rangle_n$ and $e_n = \langle n_\circ, n_\bullet \rangle_n = \langle 0, 1 \rangle_n$
- $e_n e_n = \langle n_\circ n_\circ + n_\bullet^* n_\bullet, n_\circ^* n_\bullet + n_\bullet n_\circ \rangle_n = \langle 0 + 1, 0 + 0 \rangle_n = 1$
- $e_m e_n = \langle m_\circ n_\circ + m_\bullet^* n_\bullet, m_\circ^* n_\bullet + m_\bullet n_\circ \rangle_n = \langle 0 + 0, e_m^* + 0 \rangle_n = -e_n e_m$

(also, $e_{n+1} x = \langle 0, x \rangle_{n+1}$ for all $x \in \mathbb{S}_n$). Moreover, given $i, j, k = e_{3,2}, -e_{3,1}, e_{2,1}$:

- $ii = jj = kk = ijk = -1$

C. Multivector bounds. In general, given $s \in \mathbb{S}_3$, and for $\mathcal{B} \in \{\mathcal{L}, \mathcal{R}\}$:

- $s = s_0 + (s_x e_1 + s_y e_2 + s_z e_3) + (s_{yx} e_{2,1} + s_{zx} e_{3,1} + s_{zy} e_{3,2}) + s_{zyx} e_{3,2,1}$
- $\mathcal{B}[s] \equiv \mathcal{B}_0 \cup \{\mathcal{B}_x\} \cup \{\mathcal{B}_y \cup \{\mathcal{B}_{yx}\}\} \cup \{\mathcal{B}_z \cup \{\mathcal{B}_{zx}\} \cup \{\mathcal{B}_{zy} \cup \{\mathcal{B}_{zyx}\}\}\}$ where $\mathcal{B}_i \equiv \mathcal{B}[s_i]$

and, if $s_0 + (s_x e_1 + s_y e_2 + s_z e_3) + (s_{yx} e_{2,1} + s_{zx} e_{3,1} + s_{zy} e_{3,2}) + s_{zyx} e_{3,2,1} \in \mathcal{B}[s]$, then:

- $s_0, \{s_x\}, \{s_y, \{s_{yx}\}\}, \{s_z, \{s_{zx}\}, \{s_{zy}, \{s_{zyx}\}\}\} \in \mathcal{B}[s]$

D. Functions and constants. Given $f[x]$ with $x \in X$ and $X \subseteq \mathbb{S}$:

- $f[x + \zeta] = f[x] + f^\nabla[x]\zeta$ with $\zeta \in \mathbb{S}$ such that $\zeta \neq 0$, $\zeta\zeta = 0$ and $\zeta x = x\zeta$
- $\tilde{f}[f[x]] = x$ and $\tilde{f}^\nabla[x] = (f^\nabla[\tilde{f}[x]])^{-1}$

and, given $x, j \in \mathbb{S}$ such that $jj = -1$:

- $e[x] = \rho[e_\omega[x]]$ such that $e_0[x] = 1$ and $e_{n+1}[x] = e_n[x](1 + x\varepsilon)$
- $c_h[x] = \rho[c_\omega[x]]$, $s_h[x] = \rho[s_\omega[x]]$ and $t_h[x] = \rho[s_\omega[x](c_\omega[x])^{-1}]$ where $c_\omega[x] = (e_\omega[x] + e_\omega[-x])\{0 \mid 1\}$ and $s_\omega[x] = (e_\omega[x] - e_\omega[-x])\{0 \mid 1\}$
- $c[x] = c_h[xj]$, $s[x] = -s_h[xj]j$ and $t[x] = -t_h[xj]j$
- $\omega = \{\mathbb{N} \mid\}$, $\epsilon = e[1]$ and $\pi = -\tilde{e}[-1]j$

Notes

1. The foundational element $0 = \{\mid\}$ initiates a well-founded recursion. The resulting subset of scalar elements, $\mathbb{S}_0$ (or $\mathbb{S}_{\{\}}$), along with the application to them of the arithmetic definitions in the Algebra section, reproduce the complete set of surreal numbers and operations defined by *Conway*.

2. Unlike classical surreal numbers, $\mathbb{S}$ allows bounds to contain sets of bounds (e.g., $\{a, \{b\}, \{c, \{d\}\} \mid\}$). Bound elements at different depths cannot be compared, and it is therefore interpreted that they correspond to components along different dimensions. Depth indicates the dimensionality of each component ($\{a \mid\}$ is a scalar, $\{\{b\}, \{c, \{\}\} \mid\}$ a vector, and $\{\{\{d\}\} \mid\}$ a bivector), while the maximum depth of a set indicates in which subspace the corresponding components are ($\{a \mid\}$ is in the scalar subspace, $\{\{b\} \mid\}$ in the 1D subspace, and $\{\{c\{d\}\} \mid\}$ in the 2D subspace).

3. Bound elements can be non-scalars too. During dimensional projection, pure surreal multivectors are isolated via the intersection ($\mathcal{B} \cap \mathbb{S}$) and projected directly, while nested bound sets ($\mathcal{B} \setminus \mathbb{S}$) are unboxed element by element, shifting the target dimension down step by step. For a set of strictly scalar bounds projected onto the scalar dimension $D \equiv \{\}$, this structural descent terminates through the base case $\mathcal{B}_{\{\}} \equiv (\mathcal{B} \cap \mathbb{S})_{\{\}}$, leaving the scalar bounds intact.

4. The angle bracket operator $\langle x_\circ, x_\bullet \rangle_n$ creates elements of higher dimensions by combining the projection in the lower-dimensional space, $x_\circ$, with the orthogonal complement, $x_\bullet$ factored by the new basis element, e_n. Component extraction performs the reverse operation (if $x = \langle x_\circ, x_\bullet \rangle_n$, $x_\circ = x_{n-1}$ and $x_\bullet = x_{-n}$). Although the bracket operator resembles a Cayley-Dickson doubling, the geometric product here dictates an associative *Clifford* construction, generating an algebra of positive metric ($e_1 e_1 = 1$ and $e_2 e_1 = -e_1 e_2$). Furthermore, by restricting the inverse to elements where $x_\circ x_\circ^*$ is not zero, the algebra accommodates the existence of zero divisors inherent to higher-dimensional algebras

5. The framework bypasses limits by natively incorporating the infinitesimal ε. Calculating the difference quotient and extracting the slope via the standard-part projection (or shadow function) $\rho[x]$ implements the non-standard analysis popularized by *Keisler*. Furthermore, the derivative $f^\nabla[x]$ is equivalent to the geometric derivative formalized by *Hestenes and Sobczyk*, while $f^\Delta[X]$ calculates the integral evaluating the function over boundaries extracted via infinitesimal set-subtraction ($X_\varepsilon \setminus X$ and $X \setminus X_\varepsilon$), by direct application of the generalized Stokes' Theorem.

6. The emergence of physical equations (expressed using natural units) from the geometric derivative in $\mathbb{S}_3$ aligns with the Algebra of Physical Space (APS) of *Baylis*, in order to describe relativistic, electromagnetic, and quantum phenomena entirely within the algebra of 3D space.

7. More pragmatically, numerical solutions can be found applying the algebraic structure of this framework combined with an automatic differentiation engine based on dual numbers.

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